Accurate Calculation of Control-Augmented Structural Eigenvalue Sensitivities Using Reduced-Order Models

Eli Livne*
University of California, Los Angeles, Los Angeles, California

A method is presented for generating mode shapes for model order reduction in a way that leads to accurate calculation of eigenvalue derivatives and eigenvalues for a class of control augmented structures. The method is based on treating degrees of freedom where control forces act or masses are changed in a manner analogous to that used for boundary degrees of freedom in component mode synthesis. It is especially suited for structures controlled by a small number of actuators and/or tuned by a small number of concentrated masses whose positions are predetermined. A control augmented multispan beam with closely spaced natural frequencies is used for numerical experimentation. A comparison with reduced-order eigenvalue sensitivity calculations based on the normal modes of the structure shows that the method presented produces significant improvements in accuracy.

Nomenclature

C = viscous damping matrix
$C1$, $C2$ = damping constants of viscous dampers
GC = generalized damping matrix
GK = generalized stiffness matrix
GM = generalized mass matrix
GM_F = part of the generalized mass matrix associated with the set of fictitious masses
I_{13} = fictitious moment of inertia at DOF 13
j = $\sqrt{-1}$
K = stiffness matrix
M = mass matrix
m_9 = fictitious mass at DOF 9
N = number of degrees of freedom
p = a structural parameter
s = Laplace transform variable
T = transformation matrix from physical to generalized coordinates
x = vector of physical displacements and rotations
z = vector of generalized displacements and rotations
λ = an eigenvalue
ζ = damping ratio
σ = real part of a complex eigenvalue
ω = imaginary part of a complex eigenvalue, damped frequency

Subscripts

B = set of degrees of freedom analogous to the boundary degrees of freedom of a component in substructure synthesis
I = set of degrees of freedom analogous to the interior degrees of freedom of a component in substructure synthesis
i = eigenpair sequential number

Superscript

T = transpose of a matrix

I. Introduction

LARGE, flexible lightly damped structures deployed in space must be actively controlled to suppress vibration and maintain a desired shape. In facing the problems associated with the analysis and synthesis of such systems, there is a need to integrate structural synthesis and control systems design techniques to take into account the interaction between the disciplines. Whether structural optimization and control system optimization are done sequentially or the design space is opened up to include structural and control system design variables simultaneously, it is important to efficiently and accurately calculate not only eigenvalues but also eigenvalue sensitivities with respect to system parameters.[1-3] It is also important to reduce the order of the system model used in a synthesis cycle to accelerate computation and reduce computer storage requirements.

In most applications of optimization techniques to control augmented structures, model reduction is achieved by using a small number of natural vibration modes of the structure instead of a complete finite-element model with a large number of degrees of freedom. However, it has been shown[4-6] that the derivatives of eigenvalues and response quantities with respect to structural design variables converge much more slowly than the eigenvalues and response quantities themselves when the number of modes is increased. Thus, Refs. 4-6 serve to alert the dynamics and control community to the fact that, while a certain number of natural mode shapes may be adequate to model the structural behavior properly, it may not be adequate to model the derivatives of that response with respect to design variables. Since such derivatives are essential to the optimization process, Ref. 4 concludes that the use of reduced models based on natural vibration modes may be ill advised for control-augmented structural synthesis.

In Ref. 4, it is pointed out that the problem of slow modal convergence of derivatives is likely to be more severe for large space structures with a large number of closely spaced frequencies. Using a flexible multispan beam as a representative example, Ref. 4 explores the convergence of eigenvalues and eigenvalue derivatives for reduced structural models. The beam is damped by collocated rate feedback controllers. A continuum model and three finite-element models of increasing order are used to calculate model damping ratios and their derivatives with respect to a concentrated mass. For each of the models, calculations are carried out with a full-order model and then with reduced-order models using different numbers of natural vibration modes in a truncated series.

The results show that errors in the reduced-order models derivatives of the damping ratios are about an order of magnitude larger than the errors in the damping ratios. For high-order reduced models, the accuracy of fixed- and updated-mode damping derivatives were about the same, whereas for

Received Sept. 14, 1987; revision received Aug. 30, 1988.

*Graduate Research Assistant, Department of Mechanical, Aerospace, and Nuclear Engineering.

low-order models, updated-mode damping derivatives were generally more accurate. In addition, the calculations lead to the paradoxical result that for a fixed number of natural vibration modes, the accuracy of the damping derivatives deteriorates as the finite-element model is refined. The convergence of the continuum reduced-order derivative is much slower than the finite-element derivative.

In the present paper, a method is presented for reducing a structural model so that both eigenvalues and eigenvalue derivatives are calculated with improved accuracy. The method is aimed at structures damped by a small number of concentrated forces or point-to-point dampers and whose controllability might be enhanced by the addition of concentrated masses at desired locations.

II. Eigenvalue Derivative Calculations and Substructure Synthesis Techniques in Structural Analysis

Using comparison functions in the Galerkin method, or admissible functions in its equivalent Rayleigh-Ritz method for self-adjoint systems, convergence of a truncated series of functions, chosen out of a complete set, is sought in some integral manner.[7] In most structural systems, normal mode shapes associated with the low natural frequencies of the undamped structure will usually exhibit some global patterns of displacement, while motion of a local nature appears at higher frequencies (this is not necessarily true for nearly periodic structures with weakly coupled components[8]).

When a small number of low-frequency mode shapes is used to reduce a structural model and when the dynamic behavior of the structure depends on the action of concentrated forces, it is not surprising to find large errors in the calculated eigenvalues of the modified structure. This is because modal truncation effects are likely to be more pronounced when local behavior is important.

In other methods of structural model reduction, namely, the substructure synthesis methods,[7,9,10] it is the recognition of this fact that leads to inclusion of "constraint modes" or "attachment modes" as part of the reduced basis used to approximate the motion of each substructure. These modes provide shape functions that reflect local behavior at the interface region where the components are attached to each other. Proper accounting for these effects is crucial in calculating eigenvalues of the whole structure.

There is, therefore, a similarity between the problem of reducing a structural model of a structure damped by concentrated forces and tuned by a set of concentrated masses and preparing a reduced model of a substructure to be attached to neighboring substructures through a finite number of degrees of freedom using substructure synthesis. This similarity suggests that with a better choice of shape functions for model reduction, it will be possible to improve the poor accuracy of eigenvalue derivatives predictions reported in Ref. 4, where natural vibration modes of the nominal structure were used for basis reduction.

While most previous research on substructure synthesis has focused on the natural frequencies and mode shapes of the free-vibrating, undamped structure or the complex eigenvalues of a damped, free-vibrating structural model, it is the large errors in eigenvalue sensitivities, as reported in Ref. 4, that are of primary concern in this work. Thus, there is no intention here to present a new method for substructure synthesis, but rather to examine the accuracy of reduced-order eigenvalue sensitivities when Ritz functions generated by available substructure synthesis methods are used for model reduction.

III. Method of Fictitious Masses

The method of fictitious masses, which was introduced in Ref. 11 and subsequently applied to external stores dynamic modeling in Ref. 12, is one of several techniques available for accelerating converegence in substructure coupling. It is chosen here for its simplicity of application. In this method, mode shapes of a substructure are computed while the interface points (to other substructures) are loaded with a set of concentrated masses, which are removed subsequently in formulating the equations of motion of the real structure. Mode shapes calculated in this way contain distortions at the interface region that improve the accuracy of the coupled results. These shape functions are linearly independent and they form part of a complete set. The method belongs to the class of loading interface coupling methods.[13]

It is proposed to use such a set of mode shapes to reduce the model of a structure in which several concentrated masses, point-to-point dampers, and concentrated control forces are changed in a design process. Attention is focused on discretized mathematical models with a finite number of degrees of freedom. Let the set of degrees of freedom where such changes occur be denoted B. These are analogous to the boundary degrees of freedom of substructure synthesis. Let the other degrees of freedom of the structure be denoted I (analogous to the interior degrees of freedom of a substructure). The equations of motion of the undamped free vibrating structure are then

$$\begin{bmatrix} M_{II} & M_{IB} \\ M_{BI} & M_{BB} \end{bmatrix} \begin{bmatrix} \ddot{x}_I \\ \ddot{x}_B \end{bmatrix} + \begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} \end{bmatrix} \begin{bmatrix} x_I \\ x_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

where K and M are the stiffness and mass matrices, respectively, partitioned into the I and B sets of degrees of freedom.

The equations of motion of the undamped free vibrating structure loaded with a set of fictitious masses at the B degrees of freedom are

$$\begin{bmatrix} M_{II} & M_{IB} \\ M_{BI} & M_{BB} + M_F \end{bmatrix} \begin{bmatrix} \ddot{x}_I \\ \ddot{x}_B \end{bmatrix} + \begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} \end{bmatrix} \begin{bmatrix} x_I \\ x_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{2}$$

where M_F is a diagonal mass matrix in which the nonzero (diagonal) elements represent lumped "fictitious" masses.

Assume that N_M natural frequencies and mode shapes of the mass modified system [Eq. (2)] are computed. The mode shapes matrix is partitioned as follows:

$$T = \begin{bmatrix} T_I \\ T_B \end{bmatrix} \qquad (N_I + N_B) \times N_M \tag{3}$$

where N_I and N_B are the numbers of degrees of freedom in the sets I and B, respectively.

Pre- and postmultiplication of the mass and stiffness matrices in Eq. (2) by T^T and T gives

$$GM + GM_F = \begin{bmatrix} T_I \\ T_B \end{bmatrix}^T \begin{bmatrix} M_{II} & M_{IB} \\ M_{BI} & M_{BB} + M_F \end{bmatrix} \begin{bmatrix} T_I \\ T_B \end{bmatrix} \tag{4}$$

$$GK = \begin{bmatrix} T_I \\ T_B \end{bmatrix}^T \begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} \end{bmatrix} \begin{bmatrix} T_I \\ T_B \end{bmatrix} \tag{5}$$

where it is understood that

$$GM_F = T_B^T M_F T_B \tag{6}$$

Note that the matrices $[GM + GM_F]$ and GK are diagonal generalized mass and stiffness matrices for the mass modified system, (i.e., nominal structure plus "fictitious" masses).

The equations of motion of the damped real structure are

$$\begin{bmatrix} M_{II} & M_{IB} \\ M_{BI} & M_{BB} \end{bmatrix} \begin{bmatrix} \ddot{x}_I \\ \ddot{x}_B \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C_{BB} \end{bmatrix} \begin{bmatrix} \dot{x}_I \\ \dot{x}_B \end{bmatrix} + \begin{bmatrix} K_{II} & K_{IB} \\ K_{BI} & K_{BB} \end{bmatrix} \begin{bmatrix} x_I \\ x_B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

where C_{BB} represents a $N_B \times N_B$ damping matrix due to either point-to-point viscous dampers or rate feedback forces.

The order of the structural model of Eq. (7) can be reduced using the transformation

$$\begin{bmatrix} x_I(t) \\ x_B(t) \end{bmatrix} = \begin{bmatrix} T_I \\ T_B \end{bmatrix} [z(t)] \tag{8}$$

and premultiplying by $[T_I^T T_B^T]$ to yield

$$[GM][\ddot{z}(t)] + [GC][\dot{z}(t)] + [GK][z(t)] = [0] \tag{9}$$

where

$$[GC] = T_B^T C_{BB} T_B \tag{10}$$

Taking the Laplace transform these equations lead to

$$\{s^2[GM] + s[GC] + [GK]\}[Z(s)] = [0] \tag{11}$$

Note that by using GM instead of $[GM + GM_F]$ the fictitious masses are removed from the structure. GM, GK, and GC are all $N_M \times N_M$. The matrix GK is diagonal, but GM and GC are not. When natural modes of the nominal structure are used, the only source of coupling in the equations of motion is the generalized damping matrix GC. In the present method, the generalized mass matrix GM is not diagonal and it also contributes to the coupling of the reduced-order equations. In both cases, the equations of motion are coupled. While the storage requirements are lower when using the natural modes of the nominal structure (because the generalized mass matrix is then diagonal), for most practical reduced-order models, this storage saving is relatively unimportant.

The eigenvalue problem of Eq. (11) can be solved by several possible transformations to equivalent first order $2N_M \times 2N_M$ problems of the form[7,14]

$$Ax = \lambda Dx \tag{12}$$

or by any technique for solving the quadratic eigenvalue problem directly. For a lightly damped structure, solution of the eigenvalue problem will yield $2N_M$ real and complex conjugate eigenvalues, as

$$\lambda_i = \sigma_i + j\omega_i \tag{13}$$

where σ_i and ω_i are real.

The damping ratio

$$\zeta_i = -\frac{\sigma_i}{\sqrt{\sigma_i^2 + \omega_i^2}} \tag{14}$$

is a measure of damping in a particular mode, and ω_i is the damped frequency of that mode. The root locus is constructed either directly based on (σ_i, ω_i) or given (ζ_i, ω_i). In this work, eigenvalue behavior is investigated by examining the pairs (ζ_i, ω_i). This allows direct comparison with the results of Ref. 4 where only ζ_i and $\partial\zeta_i/\partial m$ are treated. Expressions for the analytic derivative of the eigenvalue λ_i with respect to a system parameter p can be found in Refs. 1 and 4 are given in the Appendix.

In calculating the derivative of the damping ratio with the full finite-element model, full-order M, C, and K matrices along with the eigenvalues and eigenvectors of the full-order eigenvalue problem are used. When reduced models are used based on a small number of mode shapes of the structure loaded with fictitious masses, the M, C, and K matrices are replaced by the generalized matrices GM, GC, and GK [see Eqs. (4), (5) and (10)]. A fixed-modes approach is adopted here, that is, N_M fictitious mass mode shapes used for basis reduction are computed at the outset, and they are not subsequently updated.

IV. Numerical Experimentation

The multispan, flexible beam example of Ref. 4 is re-examined here (see Fig. 1). It has low-frequency, closely spaced modes. Thus, it serves to simulate a typical flexible space structure. Two massless rate controllers are placed on the beam as shown in Fig. 1. They act as viscous dampers with damping constants $C1$ and $C2$ chosen to produce damping ratios of a few percent for the first 10 modes. With this choice of a structural example for numerical experimentation, it is possible to compare results obtained from the method described herein with those reported in Ref. 4. Following Ref. 4, a list of the 10 lowest natural frequencies is presented in Table 1 as calculated with a continuum model and a 15 element finite-element model. This finite-element model has 26 unrestrained degrees of freedom (see Fig. 1b). It is accurate to better than 1.2% error in frequency for the first six modes. Attention is focused on this finite-element model to investigate convergence of eigenvalue derivatives as the number of modes used to reduce the model is increased. Errors are measured with respect to the full order 26 degree-of-freedom eigensolution. All computations in this work were carried out using the finite-element/matrix manipulation code UCLA2.[15]

Choosing "Fictitious Masses" for the Degrees of Freedom in Set B

There are many ways of selecting fictitious masses to load the "boundary" degrees of freedom to generate mode shapes for substructure coupling. References 11 and 12 recommend using masses and moments of inertia proportional to the forces and moments transferred through the interface and of the same order as the mass of the substructure to be attached. However, no substructures are attached to the beam here, and some other guidelines are needed for the determination of the proper size of fictitious masses for degrees of freedom where controllers act or masses are changed. In Ref. 11, it was shown that when large masses are used, the results are insensitive to the size of the masses as long as ill conditioning is not encountered. Thus, for example, large multiples of mass and moment

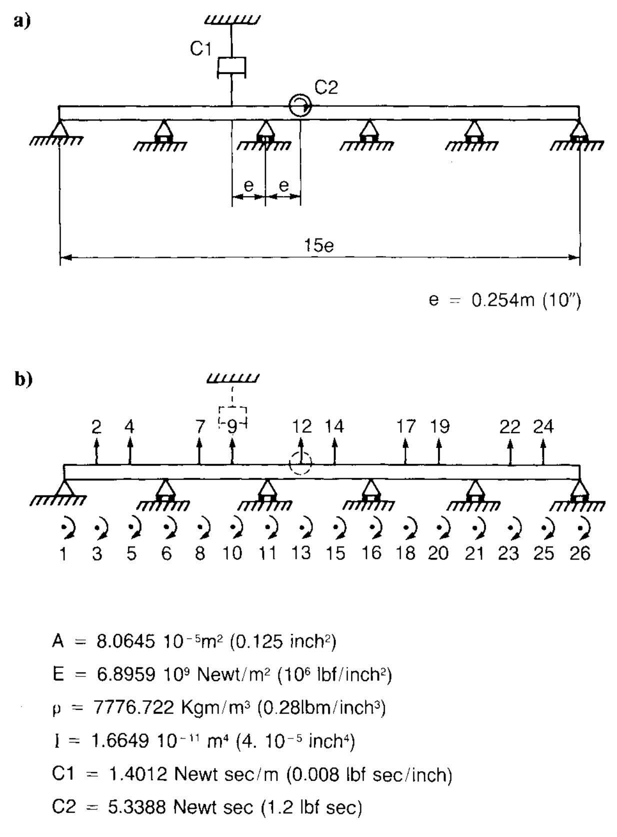

Fig. 1 Multispan beam geometry and unconstrained degrees of freedom.

of inertia of a single element might be used, as indeed is done here. When an alternative set of fictitious mass and moment of inertia values for degrees of freedom 9 and 13 was selected, namely, the mass and moment of inertia of the whole beam, the results were very similar.

Following Ref. 4, damping ratios and their sensitivities with respect to mass at degree of freedom 9 (Fig. 1b) are first considered here. Set B includes degrees of freedom 9 and 13 (see Fig. 1b) where the control forces are applied. Fictitious masses are added to the degrees of freedom in set B. A modified structure is thus created. A subset of the natural mode shapes corresponding to the modified structure serves as a reduced basis. A 12-mode reduced model was used. Reference values of m_9 (mass at DOF 9) and I_{13} (mass moment of inertia at DOF 13) were selected to be the lumped mass and inertia of a single-beam element, namely, $m_9 = 0.15873$ kgm (0.35 lbm) and $I_{13} = 8.53396\ 10^{-4}$ kgm $\cdot$ m^2 (2.9167 lbm-in^2).

Figures 2 and 3 are plots of the percent of error vs the size of fictitious masses used. For all values of fictitious masses used, accuracy of the damping and damping derivative with respect to m_9 show an improvement over the results of model reduction using natural modes of the nominal structure. It also is observed that above a multiple of 50 times the reference values, the error stabilizes and becomes insensitive to the size of the masses used. This leaves us with the freedom to those any size of fictitious masses as long as they are large in comparison with typical local masses, but not large enough to cause ill conditioning. For the following calculations, the fictitious mass values were conservatively selected as the reference values multiplied by 500.

Convergence

Modes of the nominal structure plus fictitious masses at DOF 9 and 13 are now used to reduce the order of the 26 DOF model. Using the full-order finite-element model (26 DOF) as the reference solution, percentage errors for damping ratios and their derivatives with respect to m_9 are plotted in Figs. 4 and 5 for the second and fourth mode, respectively. Corresponding error plots for damped frequency and its derivative with respect to m_9 are shown in Figs. 6 and 7. With a 12-mode reduced-order model based on fictitious mass mode shapes errors in damping ratio drop to less than 0.1% in modes 1–5 and less than 5% in modes 6 and 7.

The error in the derivative of the damping ratio with respect to m_9 drops to less than 0.35% in modes 1–5 and less than 12% in modes 6 and 7. Compared with the error when using the natural modes of the nominal structure,[4] this is a very significant improvement in accuracy.

Examine the results for the fourth mode (Fig. 5). Using natural modes of the nominal structure the convergence of the damping ratio derivative is very slow. With 24 out of the full set of 26 modes in the reduced model, Ref. 4 reports 36% error in the derivative. With 12 modes in the reduced model, the derivative error is about 900% and the damping error is 2.9%. With 12 modes of the beam with fictitious masses, the error in the derivative is 0.35% and the error in damping ratio is 0.032%.

Figures 6 and 7 show improvement in accuracy for damped frequency and damped frequency derivative when modes of the structure with fictitious masses are used rather than modes of the nominal structure. Among the first six modes, the highest errors in damped frequency derivative and damping ratio appear in mode 6 (see Tables 2 and 3 in the column corresponding to fictitious masses at DOF 9 and 13). As Table 1 shows, a first frequency band, 1–2.5 Hz including modes 1–5, can be identified in the spectrum of beam natural frequencies. Indeed, for modes 1–5, high accuracy in damping ratio, frequency, and their derivatives is achieved with even nine "fictitious mass modes" in the reduced model. Mode 6 belongs to the next frequency band and is very close to mode 7. It is because of this that the convergence rates of $\partial\zeta/\partial m_9$ and $\partial\omega/\partial m_9$ for the sixth mode are slower. Nevertheless, with "fictitious mass modes," the accuracy of reduced model results is markedly improved.

It is interesting to notice the relatively large reduced-order model errors in derivatives of damped frequencies when natural modes of the nominal structure are used. In an undamped structure, use of natural modes for order reduction leads to reduced-order frequencies and frequency derivatives identical to those of the full-order model. Intuition might suggest that

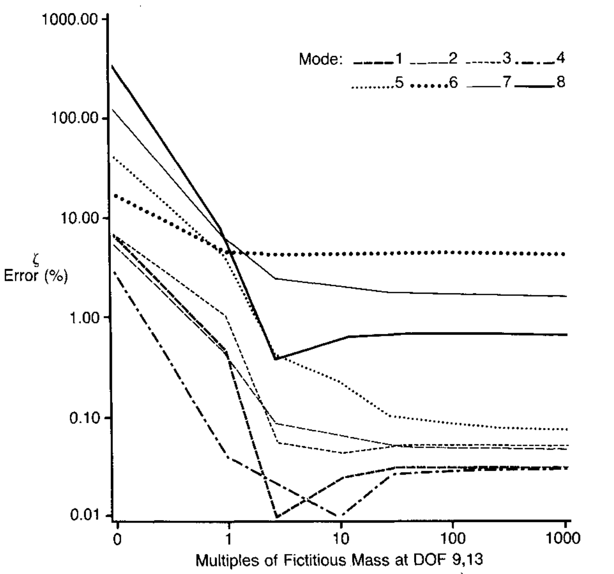

Fig. 2 Modal damping ratio error with a 12-mode fictitious mass reduced-order model.

Table 1 Natural frequencies of the multispan beam, Hz

Mode	Continuum solution[4]	UCLA2[15]
1	1.159823	1.160312
2	1.286765	1.287547
3	1.609089	1.610961
4	2.026769	2.030946
5	2.433312	2.440912
6	4.639294	4.692309
7	4.904000	4.966080
8	5.511752	5.597137
9	6.242831	6.358532
10	6.925819	7.064366

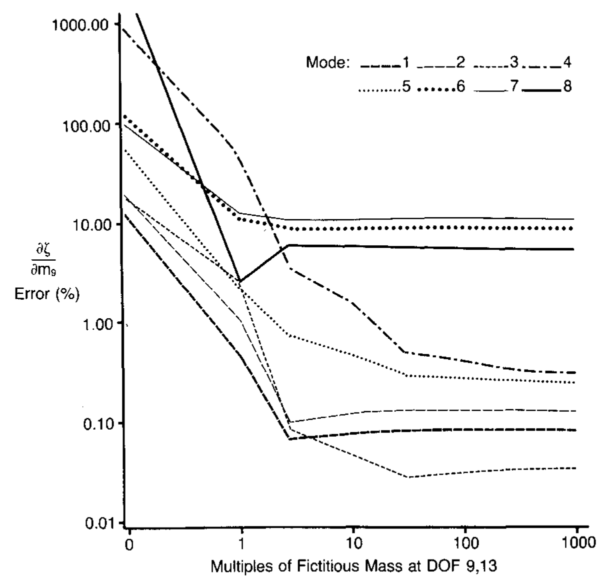

Fig. 3 Modal damping ratio derivative error with a 12-mode fictitious mass reduced-order model.

Table 2 Percent of damping ratio ζ errors with different B sets (12-mode reduced model)

Mode	Fict. mass at DOF 9	Fict. mass at DOF 13	Fict. mass at DOF 9 and 13
1	6.671	0.103	0.032
2	5.897	0.113	0.050
3	8.549	0.095	0.056
4	4.330	0.009	0.031
5	37.949	0.031	0.079
6	7.002	0.437	4.663
7	96.706	0.210	1.775
8	296.580	0.315	0.723

Table 3 Percent of damping ratio sensitivity errors ($\partial\zeta/\partial m_9$) with different B sets (12-mode reduced model)

Mode	Fict. mass at DOF 9	Fict. mass at DOF 13	Fict. mass at DOF 9 and 13
1	9.363	0.538	0.087
2	16.892	0.055	0.139
3	16.706	0.546	0.035
4	904.144	11.830	0.347
5	69.698	1.162	0.272
6	127.264	1.430	9.812
7	90.966	4.359	12.174
8	1887.640	0.855	5.936

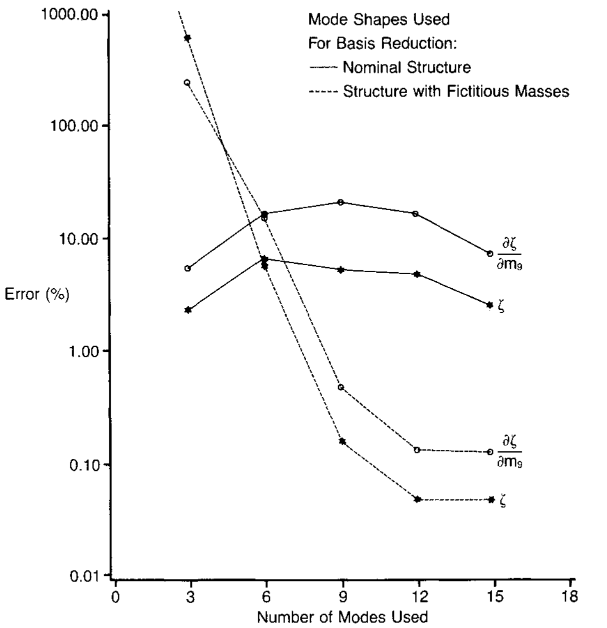

Fig. 4 Errors in damping ratios and their derivatives—mode 2.

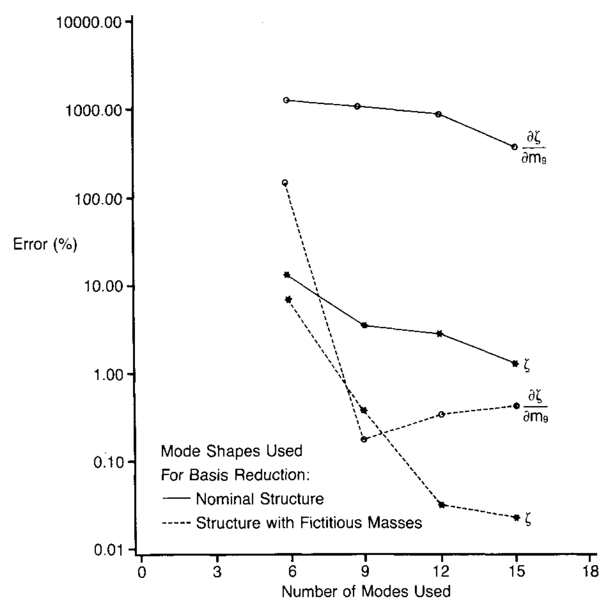

Fig. 5 Errors in damping ratios and their derivatives—mode 4.

in a lightly damped structure, good accuracy of frequency derivative still might be achieved using natural modes of the structure for order reduction. The results of Figs. 6 and 7, however, indicate that this is not the case.

Selecting Degrees of Freedom for Set B

Given a space structure with dampers (or control forces) acting at known degrees of freedom and concentrated masses to be placed at predetermined locations to enhance controllability, it is important to consider the question of how to select set B before obtaining fictitious mass modes for basis reduction. Tables 2 and 3 present errors in the damping ratio ζ and damping ratio derivative $\partial\zeta/\partial m_9$ for 12-mode reduced models based on different B sets for the multispan beam.

Examination of the results presented in Tables 2 and 3 indicated that 1) adding a fictitious mass only at DOF 9 leads to unacceptably high errors; 2) adding a fictitious mass moment of inertia only at DOF 13 leads to major improvements in accuracy relative to the results reported in Ref. 4; and 3) adding fictitious masses at DOF 9 and 13 leads to high-quality results for the first five damping ratios and their derivatives and manageable errors for the next three damping ratios and their derivatives.

V. Damping of a Variable Mass Space Structure

In the preceding sections, the mathematical model of a damped structure was reduced in order using fictitious mass modes. A set B of degrees of freedom was chosen to contain the degrees of freedom acted upon by the damping control forces. Thus, a better description of the mechanism of energy dissipation due to local action is available. The "fictitious mass" approach to improved accuracy for reduced-order dynamic models also may be useful in dealing with two other problems that are likely to arise in the design of large space structures.

1) In many cases of practical interest, large space structures may be subject to changes in mass while in service. The docking and undocking of spaceships and the changing mass of reservoirs and storage centers are just a few examples. An active vibration suppression system designed for one mass distribution is not necessarily adequate for others, and in design optimization of such a system, care must exercised to consider several alternative mass distributions.

2) When a design synthesis of a control-augmented structure is carried out by solving a sequence of approximate problems,[16] there will be changes in mass design variables in each cycle. If the natural modes of the current design are used for order reduction, it is usually necessary to update them as the structure changes.

In both cases it is desirable that one set of shape functions for model order reduction be suitable for the analysis of structures whose differences are confined to some local areas.

Consider a space structure controlled by a relatively small number of concentrated control forces and balast masses. In-service mass changes are assumed to be in the form of addition or subtraction of concentrated masses. If the location of dampers and variable concentrated masses are known, the proper choice of the B set of degrees of freedom will enable successful order reduction. Let fictitious masses be placed at degrees of freedom corresponding to control force action,

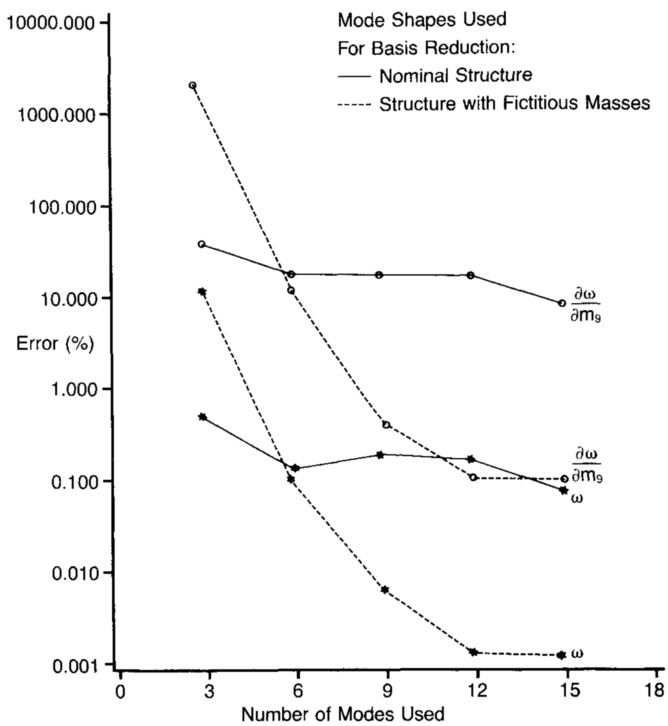

Fig. 6 Errors in damped frequencies and their derivatives—mode 2.

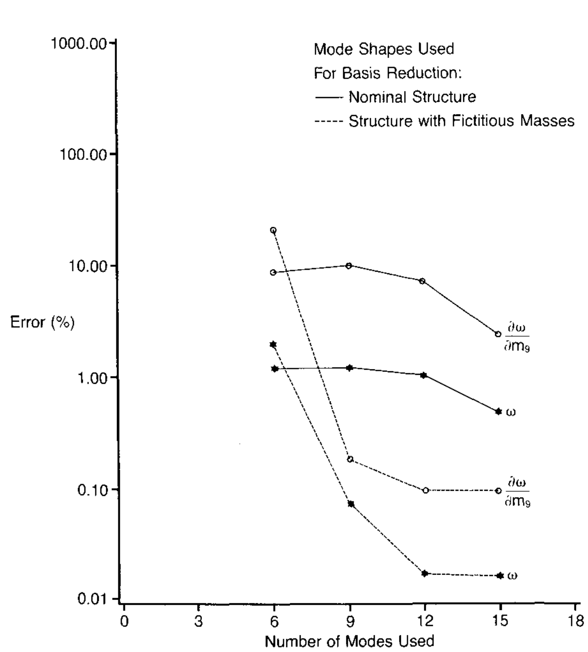

Fig. 7 Errors in damped frequencies and their derivatives—mode 4.

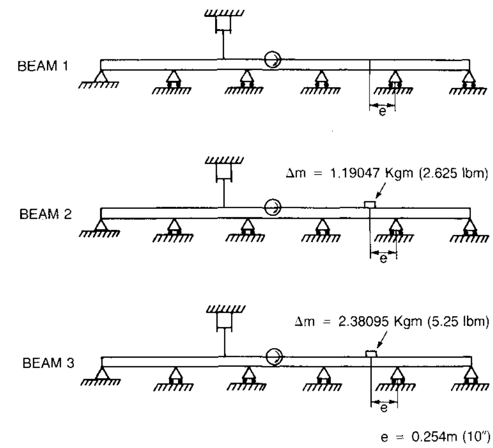

Fig. 8 Beam with a variable size mass at DOF 19.

Table 4 Percent of damping ratio ζ errors for multispan beam with three different mass distributions using the same 12 modes of the basic structure with fictitious masses to reduce the model[a]

	Fict. masses at DOF 9 and 13			Fict. masses at DOF 9, 13 and 19		
Mode	Beam 1	Beam 2	Beam 3	Beam 1	Beam 2	Beam 3
1	0.031	0.547	0.650	0.018	0.003	0.002
2	0.050	0.074	0.073	0.117	0.026	0.025
3	0.056	2.864	3.957	0.086	0.015	0.025
4	0.031	0.065	0.008	0.189	0.106	0.102
5	0.079	0.053	0.042	0.049	0.020	0.019
6	4.663	5.065	6.630	2.725	0.516	0.549
7	1.775	0.664	0.690	1.311	0.356	0.369
8	0.723	8.511	9.614	6.115	1.251	1.093

[a]Beam 1 = nominal beam of Fig. 1, Beam 2 = beam 1 + 50% of its weight added to DOF 19, Beam 3 = beam 1 + 100% of its weight added to DOF 19. Numbers rounded to three decimal places.

Table 5 Percent of damping ratio derivative $\partial\zeta/\partial m_{19}$ errors for multispan beam with three different mass distributions using the same 12 modes of the basic structure with fictitious masses[a]

	Fict. masses at DOF 9 and 13			Fict. masses at DOF 9, 13 and 19		
Mode	Beam 1	Beam 2	Beam 3	Beam 1	Beam 2	Beam 3
1	0.124	0.255	0.481	0.114	0.013	0.007
2	0.127	0.040	0.024	0.120	0.007	0.005
3	0.229	1.655	2.237	0.204	0.002	0.020
4	0.292	1.687	2.329	0.448	0.010	0.025
5	0.240	0.762	1.220	0.392	0.077	0.070
6	13.880	0.776	0.261	15.600	0.339	0.445
7	16.900	3.963	3.363	18.640	1.984	1.779
8	0.115	4.097	4.723	4.891	2.077	1.688

[a]Beam 1 = nominal beam of Fig. 1, Beam 2 = beam 1 + 50% of its weight added to DOF 19, Beam 3 = beam 1 + 100% of its weight added to DOF 19. Numbers rounded to three decimal places.

mass balance, and variable service mass. A set of mode shapes is calculated and used to reduce the structural model. This same set of modes then is used for model reduction in all service mass combinations.

To demonstrate the power of the method, three derivatives of the multispan beam were used. See Fig. 8. The three beams differ in mass at DOF 19. Beam 1 is the nominal beam of Fig. 1. Beam 2 is beam 1 with an added concentrated mass at DOF 19. The added mass in beam 2 is 50% of the total mass of the original beam. In beam 3, the added mass at DOF 19 is 100% the mass of the original beam. The same set of dampers as in Fig. 1 was used.

Damped frequencies ω, damping ratios ζ and their derivatives with respect to the mass at DOF 19, $\partial\zeta/\partial m_{19}$ and $\partial\omega/\partial m_{19}$ were calculated with full-order (26 DOF) models of the three beams and 12-mode reduced-order models based on the nominal beam with fictitious masses.

With fictitious masses at DOF 9 and 13 (damper action) and 19 (variable mass), high accuracy of the reduced-order damping ratios and their derivatives is achieved for all three beams using the same set of modes. See Tables 4 and 5. The accuracy of the damped frequency and its derivative is presented in Tables 6 and 7. With a fictitious mass placed at DOF 19 (where the mass change takes place) in addition to DOF 9 and 13 (damper locations), better overall accuracy is achieved for beams 2 and 3, while the overall accuracy for beam 1 is not improved significantly.

If only one mass distribution is considered and a concentrated mass at DOF 19 is changed as a design variable during synthesis of a control augmented beam, the results show that

Table 6 Percent of frequency ω errors for multispan beam with three different mass distributions using the same 12 modes of the basic structure with fictitious masses[a]

	Fict. masses at DOF 9 and 13			Fict. masses at DOF 9, 13 and 19		
Mode	Beam 1	Beam 2	Beam 3	Beam 1	Beam 2	Beam 3
1	0.005	0.142	0.188	0.008	0.002	0.000
2	0.001	0.009	0.010	0.009	0.007	0.008
3	0.003	0.050	0.052	0.002	0.000	0.000
4	0.016	0.035	0.033	0.048	0.006	0.005
5	0.028	0.077	0.081	0.051	0.043	0.044
6	0.019	0.368	0.407	0.147	0.007	0.003
7	0.197	0.179	0.182	0.165	0.185	0.187
8	0.014	0.139	0.143	0.143	0.002	0.001

[a]Beam 1 = nominal beam of Fig. 1, Beam 2 = beam 1 + 50% of its weight added to DOF 19, Beam 3 = beam 1 + 100% of its weight added to DOF 19. Numbers rounded to three decimal places.

Table 7 Percent of frequency derivative $\partial\omega/\partial m_{19}$ errors for the multispan beam with three different mass distributions using the same 12 modes of the basic structure with fictitious masses[a]

	Fict. masses at DOF 9 and 13			Fict. masses at DOF 9, 13 and 19		
Mode	Beam 1	Beam 2	Beam 3	Beam 1	Beam 2	Beam 3
1	0.112	0.067	0.062	0.085	0.008	0.003
2	0.027	1.273	0.986	0.006	0.250	0.201
3	0.185	0.207	0.119	0.147	0.036	0.019
4	0.178	0.150	0.345	0.300	0.178	0.133
5	0.421	1.054	0.963	0.496	0.092	0.124
6	2.811	2.575	2.377	0.171	0.336	0.187
7	12.051	25.895	21.376	1.519	16.250	12.969
8	1.106	1.426	1.142	1.336	0.481	0.228

[a]Beam 1 = nominal beam of Fig. 1, Beam 2 = beam 1 + 50% of its weight added to DOF 19, Beam 3 = beam 1 + 100% of its weight added to DOF 19. Numbers rounded to three decimal places.

the same set of modes can be used throughout the synthesis, even when a large variation of m_{19} has taken place. No updates of the modes used for basis reduction are needed.

Thus, it is recommended to include in the degrees of freedom of the set B (for control-augmented structural synthesis applications) degrees of freedom where mass is changed (to ensure proper single-basis reduction for the different mass distributions checked in the course of optimization) and degrees of freedom where concentrated control forces act (to take account of the localized nature of energy dissipation mechanism).

VI. Other Methods of Dynamic Substructure Synthesis

It has been shown that the "fictitious mass" technique of component mode synthesis can be used for order reduction in control-augmented structural systems. Other methods of generating reduced-order bases in substructure synthesis can also be adapted to this purpose successfully. For a review of current modal synthesis methods, the reader may consult Ref. 17. For the multispan beam, a parallel set of model order-reduction calculations was carried out using the popular Craig-Bampton technique,[18] which yielded practically the same results as those presented here.

This is not surprising. As Ref. 19 points out, both the fixed and free interface variants of the modal synthesis method are limiting cases of the more general version with a loaded interface. For the multispan beam treated here, modes of the beam with fictitious masses at the B degrees of freedom, analogous to the boundary degrees of freedom of a substructure in substructure synthesis, can be divided into two groups when the "fictitious masses" are very large. One group contains modes of very low frequency that are practically the modes of the structure with loading masses at the B degrees of freedom only and no other mass. As the fictitious masses increase, the frequencies of these modes decrease and they tend to span the same subspace as the constraint modes of the Craig-Bampton method. In the second group of modes, the degrees of freedom loaded by the large fictitious masses show very small displacements and rotations. These modes approach the corresponding fixed interface normal modes of the Craig-Bampton method in shape and frequency. As Ref. 19 points out, "a fixed interface is conveniently simulated by adding an infinite mass to it, to hold it still." Thus, in the case analyzed here, as the fictitious masses increase, the fictitious masses method results approach those of the Craig-Bampton method. This explains the convergence of results shown in Figs. 2 and 3, as well as the insensitivity to the size of fictitious masses used when they are large.

It may be argued that in the selection of values for fictitious masses in the fictitious mass method, there is an undesirable element of "art." However, as Ref. 11 shows for undamped frequencies and as this work demonstrates for damping ratios, damped frequencies, and their derivatives, choosing the fictitious masses to be large enough (100–500 times a typical mass in the region of concern) ensures high accuracy of the reduced-order model calculations based on fictitious mass modes. Thus, there is no need for a lot of numerical experimentation before choosing the size of masses and alternative selections can be made with very similar results.

In the Craig-Bampton method, both a static analysis and an eigenvalue analysis of the structure with different boundary conditions are needed. In the fictitious mass method, the mode shapes needed for model reduction are generated by one eigenvalue analysis of the structure with modifications of the diagonal elements of the mass matrix that are very simple to incorporate. This convenient feature of the fictitious mass method makes it attractive in the context of the eigenvalue sensitivity analysis presented.

VII. Concluding Remarks

Using natural vibration modes of a structure to obtain a reduced-order model for active vibration suppression purposes leads to slow convergence of the derivatives of damping ratios and damped frequencies to their full-order finite-element values. When a small number of such modes is used, the error in derivatives can be very high. Moreover, in a synthesis process, these modes must be updated often as the structure changes.

In the present work, a method is presented for generating mode shapes for model reduction in a way that provides good convergence of damping ratios, damped frequencies, and their derivatives to their full-order values. This is done by treating degrees of freedom where control forces act or masses are changed in a manner analogous to that used for boundary degrees of freedom in component mode synthesis. Large fictitious masses are placed at these "boundary" degrees of freedom. The natural modes of the modified structure then are used to reduce the order of the mathematical model of the original structure. In the order reduction process, the fictitious masses are removed.

It has been demonstrated numerically that using fictitious mass mode shapes for basis reduction significantly improves the accuracy of reduced-order eigenvalue and eigenvalue derivatives. Alternatively, a reduced basis generated by the Craig-Bampton method can be used for model order reduction with essentially the same results.

A multispan, flexible beam controlled by direct-rate feedback is used to simulate a space structure with low, closely spaced natural frequencies. Based on the results reported herein, it seems that, contrary to the conclusions of Ref. 4, it is practical and accurate to use model reduction for calculating derivatives of the damping ratios and damped frequencies with respect to structural and control parameters (even when control forces are concentrated or act locally and concentrated masses are used to enhance controllability). The technique presented herein is useful when the locations of controllers and variable masses are predetermined and it is particularly efficient for reducing the order of control-augmented structures when the number of point-to-point controllers or variable concentrated masses is small.

Appendix

The derivative of the eigenvalue λ_i with respect to a system parameter p is given by[4]

$$\frac{\partial \lambda_i}{\partial p} = -\frac{\phi_i^T[\lambda_i^2(\partial M/\partial p) + \lambda_i(\partial C/\partial p) + (\partial K/\partial p)]\phi_i}{\phi_i^T[2\lambda_i M + C)\phi_i} \tag{A1}$$

where $\{\phi_i\}$ is the eigenvector corresponding to λ_i.

Separated to real and imaginary parts, Eq. (A1) becomes

$$\frac{\partial \lambda_i}{\partial p} = \frac{\partial \sigma_i}{\partial p} + j\frac{\partial \omega_i}{\partial p} \tag{A2}$$

Differentiation of Eq. (14) leads to the derivative of the damping ratio as

$$\frac{\partial \zeta_i}{\partial p} = \frac{\omega_i[\sigma_i(\partial \omega_i/\partial p) - \omega_i(\partial \sigma_i/\partial p)]}{[\sigma_i^2 + \omega_i^2]^{3/2}} \tag{A3}$$

Acknowledgments

This research was supported by NASA Research Grant NSG 1490. The interest and support of Prof. L. A. Schmit Jr. are greatly appreciated.

References

[1]Haftka, R. T., Martinovic, Z. N., and Hallauer, W. L., "Enhanced Vibration Controllability by Minor Structural Modification," *AIAA Journal,* Vol. 23, Aug. 1985, pp. 1260–1266.

[2]Haftka, R. T., Martinovic, Z. N., Hallauer, W. L., Jr., and Schamel, G., "Sensitivity of Optimized Control Systems to Minor Structural Modifications," AIAA Paper 85-0807, April 1985.

[3]Bodden, D. S. and Junkins, J. L., "Eigenvalue Optimization Algorithms for Structure/Controller Design Iteration," *Journal of Guidance and Control,* Vol. 8, Nov.–Dec. 1985, pp. 697–706.

[4]Sandridge, C. A. and Haftka, R. T., "Accuracy of Derivatives of Control Performance Using a Reduced Structural Model," AIAA Paper 87-0905, April 1987.

[5]Haftka, R. T. and Kamat, M. P., *Elements of Structural Optimization,* Vol. Martinus Nijhoff, Dordrecht, the Netherlands, 1985, pp. 172–173.

[6]Haftka, R. T. and Yates, E. C., Jr., "Repetitive Flutter Calculations in Structural Design," *Journal of Aircraft,* Vol. 13, July 1976, pp. 454–461.

[7]Meirovich L., *Computational Methods in Structural Dynamics,* Sijhoff & Noordhoff, Alphen aan den Rijn, the Netherlands, 1980, Chap. 11.

[8]Pierre, C., "Localized Free and Forced Vibrations of Nearly Periodic Disordered Structures," AIAA Paper 87-0774, April 1987.

[9]Hintz, R. M., "Analytical Methods in Component Modal Synthesis," *AIAA Journal,* Vol. 13, Aug. 1975, pp. 1007–1016.

[10]Craig, R. R., Jr., *Structural Dynamics—An Introduction to Computer Methods,* Wiley, New York, 1981, Chap. 19.

[11]Karpel, M. and Newman, M., "Accelerated Convergence for Vibration Modes using the Substructure Coupling Method and Fictitious Coupling Masses," *Israel Journal of Technology,* Vol. 13, May 1975, pp. 55–62.

[12]Karpel, M., "Efficient Vibration Mode Analysis of Aircraft with Multiple External Stores Configurations," *Journal of Aircraft,* Vol. 25, Aug. 1988, pp. 747–751.

[13]Benfield, W. A. and Hruda, R. F., "Vibration Analysis of Structures by Component Mode Substitution," *AIAA Journal,* Vol. 9, July 1971, pp. 1255–1261.

[14]Laub, A. J. and Arnold, W. F., "Controllability and Observability Criteria for Multivariable Linear Second Order Models," *IEEE Transactions on Automatic Control,* Vol. AC-29, Feb. 1984, pp. 163–165.

[15]Dong, S. B., "UCLA—Unified Computer Language for Analysis," School of Engineering and Applied Science, Civil Engineering Dept., University of California, Los Angeles, 1984.

[16]Lust, R. V. and Schmit, L. A., "Control-Augmented Structural Synthesis," *AIAA Journal,* Vol. 26, Jan. 1988, pp. 86–95.

[17]Craig, R. R., Jr., "A Review of Time Domain and Frequency Domain Component Synthesis Methods," *Combined Experimental/Analytical Modeling of Dynamic Structural Systems,* ASME, Applied Mechanics Division, AMD-Vol. 67, 1985, pp. 1–30.

[18]Craig, R. R., Jr. and Bampton, M. C. C., "Coupling of Substructures for Dynamic Analysis," *AIAA Journal,* Vol. 6, July 1968, pp. 1313–1319.

[19]Curnier, A., "On Three Modal Synthesis Variants," *Journal of Sound and Vibration,* Vol. 90, No. 4, 1983, pp. 327–340.